

# No-compressing of quantum phase information

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(Dated: January 25, 2013)

We raise a general question of quantum information theory whether the quantum phase information can be compressed and retrieved. A general qubit contains both amplitude and phase information, while an equatorial qubit contains only a phase information. We study whether it is possible to compress the phase information of  $n$  equatorial qubits into  $m$  general qubits with  $m$  being less than  $n$ , and still those information can be retrieved perfectly. We prove that this process is not allowed by quantum mechanics.

PACS numbers: 03.65.Ta, 03.67.Ac, 03.65.Aa, 03.67.Lx

In quantum information processing, we can perform a lot of miraculous tasks by using the principles of quantum mechanics, such as, we can teleport an unknown quantum state [1] by using Einstein-Podolsky-Rosen (EPR) pair [2], we can have a quantum computer which surpasses its classical counterpart [3], we can construct protocols of quantum key distribution with unconditional security [4, 5], etc. On the other hand, some tasks are not allowed by quantum mechanics, for example, quantum information can not be cloned perfectly [6], it is even not allowed by quantum mechanics to delete an unknown quantum state [7], it is impossible to design a secure protocol of quantum bit commitment [8, 9], the superluminal communication is forbidden [10], etc.

It is continuously of broad interest and fundamental to explore the realm of quantum mechanics to find what is possible and what is impossible. Let us start with a problem of classical case: a sequence of 10 bit can encode  $2^{10} = 1024$  different information, while if the number of zeros and ones are equal in this bit sequence, it can only encode  $10!/(5!5!) = 252$  different information. Then the information represented in this special 10 bit sequence can be compressed into a general 8 bit sequence, since  $2^8 = 256 > 252$ . Next let us consider the case of quantum information. A qubit contains both amplitude and phase information which can be represented explicitly in a Bloch sphere by two angles  $\theta$  and  $\phi$ ,

$$|\varphi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle, \quad (1)$$

where  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi)$ . If we consider a specified qubit located in the equator of the Bloch sphere, we have an equatorial qubit,

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi} |1\rangle), \quad (2)$$

there is only one phase parameter  $\phi$ . Of course, we know that by a unitary transformation similar to the Hadamard gate,  $U_r$  which will be presented later in (5) and its inverse  $U_r^\dagger$ ,

the phase parameter and the amplitude parameter can be exchanged. A general qubit and an equatorial qubit are different. For example, in universal quantum cloning, the optimal fidelity for qubit can be around 83.3%, [11], while for equatorial qubit, the optimal fidelity can be higher and achieves about 85.4%, [12, 13].

We can have another example: suppose we would like to teleport an unknown two-qubit state but with a partially known form,  $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ . Instead of using two EPR pairs, we can *extract* the information in this two-qubit state by a local controlled-NOT (CNOT) gate applied on  $|\Psi\rangle$  with the first qubit as the controlled qubit and the second as the target gate. Alice, the sender, then obtains,  $|\Psi_A\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle$ . Alice only needs to teleport the first qubit by a resource of one EPR pair [1], Bob, the receiver, would receive  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  by teleportation. By adding an ancillary state  $|0\rangle$ , Bob can locally perform a CNOT gate and finally recover the original two-qubit state,  $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ . So we are able to *compress* the information of a partially known two-qubit state to a single qubit, such that we can teleport it by a resource of only one EPR pair.

The fact that we can teleport a two-qubit state,  $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ , by only one EPR pair is due to that this two-qubit state, or we can rewrite it as,  $|\Psi\rangle = \cos \frac{\theta}{2} |00\rangle + \sin \frac{\theta}{2} e^{i\phi} |11\rangle$ , contains only the information of two angles, namely  $\theta$  and  $\phi$ , just as an ordinary qubit  $|\varphi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle$  does. We may then ask a question in an opposite direction: since each equatorial qubit has a fixed  $\theta = \pi/2$ , and contains only the information of a single angle,  $\phi$ , is it possible to *compress* the information of two angles of two equatorial qubits into just one general qubit? If yes, we can teleport it by only one EPR pair, then can we *separate* the information from this qubit into two equatorial qubits and recover their original form (We assume that only reversible operations are used in the recovery process, hence measurement is not allowed)? We will next prove that this procedure is not allowed by quantum mechanics!

Let us start with a simple example. Consider two equatorial

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qubits as the following,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi_1}|1\rangle) \quad (3)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi_2}|1\rangle) \quad (4)$$

Naturally, we might rotate one of the equatorial qubits so that the equatorial large circle would be rotated to a longitudinal one by unitary transformation,

$$U_r = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}, \quad (5)$$

so we have,

$$U_r|\psi_1\rangle = \sin\frac{\phi_1}{2}|0\rangle + \cos\frac{\phi_1}{2}|1\rangle. \quad (6)$$

Now, the angle of phase information is changed to the angle of amplitude information. For the two qubits,  $(U_r|\psi_1\rangle)|\psi_2\rangle$ , apply a CNOT gate to the system with the first qubit as the controlled qubit and the second as the target gate, and we would obtain,

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[(\cos\frac{\phi_1}{2}|1\rangle + \sin\frac{\phi_1}{2}e^{i\phi_2}|0\rangle)|1\rangle + (\sin\frac{\phi_1}{2}|0\rangle + \cos\frac{\phi_1}{2}e^{i\phi_2}|1\rangle)|0\rangle]. \quad (7)$$

It is now clear, by measuring the second qubit, for both  $|0\rangle$  and  $|1\rangle$  cases, we would get a single qubit containing the information of both  $\phi_1$  and  $\phi_2$ ,

$$|\varphi_1\rangle = \cos\frac{\phi_1}{2}|1\rangle + \sin\frac{\phi_1}{2}e^{i\phi_2}|0\rangle, \quad (8)$$

$$|\varphi_2\rangle = \sin\frac{\phi_1}{2}|0\rangle + \cos\frac{\phi_1}{2}e^{i\phi_2}|1\rangle. \quad (9)$$

It is now possible to teleport one qubit with two angles information by one EPR pair. Then it is Bob's problem: is it possible to separate the phase and amplitude information from  $|\varphi_{1(2)}\rangle$ ? However, for example in case,  $|\varphi_1\rangle$ , Bob would find that if  $\phi_1 = 0, \pi$ , the phase information in  $|\varphi_1\rangle$  would be totally lost. Then we conclude that it is impossible to recover the original two equatorial qubits. This procedure is represented in Fig. 1. We shall provide a proof later for the general case.

We may also understand this in the language of topology. The initial state constructs a space that is the direct product of two one-dimensional rings  $S^1 \times S^1$ . And this space is homeomorphic to the ordinary torus surface in our three-dimensional space. However, the state,  $|\varphi_1\rangle = \cos\frac{\phi_1}{2}|1\rangle + \sin\frac{\phi_1}{2}e^{i\phi_2}|0\rangle$ , forms a two-dimensional sphere  $S^2$ . Since these two spaces are not homeomorphic, there exists no bijective continuous mapping from one space to another.

Now let us consider a general case. One may find that the problem of compressing and retrieving of quantum phase information is whether it is possible by unitary transformation to retrieve the information of  $m$  qubits into  $n$  equatorial qubits,

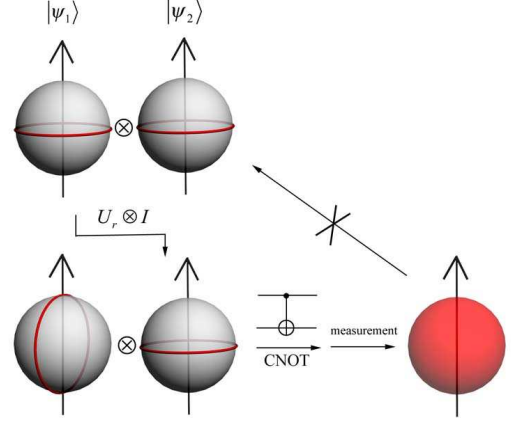


FIG. 1: (color online) An equatorial qubit is located in the equator of the Bloch sphere. Two equatorial qubits by one single qubit rotation  $U_r$  and a CNOT gate followed by a measurement can lead to a qubit with two angles information. However, it is impossible to recover the original form of two equatorial qubits from just one single qubit with both phase and amplitude information.

$n > m$ . Or similarly, whether a unitary transformation can change  $n$  equatorial qubits to  $m$  qubits,

$$U \bigotimes_{k=1}^n |\psi_k\rangle |A\rangle = \bigotimes_{k=1}^m |\varphi_k\rangle |B\rangle, \quad (10)$$

where,  $|\psi_k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi_k}|1\rangle)$ , is the equatorial qubit, and  $|\varphi_k\rangle$  is a general qubit,  $|A\rangle, |B\rangle$  are ancillary states. We can let  $\phi_k = 0, \pi$ , so we have  $|\psi_k\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ , there are altogether  $2^n$  orthogonal states, while  $m$  general qubits can provide  $2^m$  orthogonal states. We know that unitary transformations keep orthogonal states to be orthogonal, when  $n > m$ , the unitary operator  $U$  does not exist. Thus the procedure of compressing and retrieving quantum phase information is impossible.

For a more general  $d$ -dimension case, we also have a similar result. Define the  $d$ -dimension equatorial state as,  $\sum_{j=0}^{d-1} e^{i\phi_{kj}} |j\rangle$ , where the normalization factor  $\frac{1}{\sqrt{d}}$  is omitted hereafter. It forms a  $(d-1)$ -dimensional torus  $T^{d-1} = S^1 \times \dots \times S^1$ , the direct product of  $(d-1)$  one-dimensional rings. The  $n$  input states can be expressed as,

$$\psi_{in} = \bigotimes_{k=1}^n \left( \sum_{j=0}^{d-1} e^{i\phi_{kj}} |j\rangle \right). \quad (11)$$

The question is, whether there exists a unitary transformation,  $U$ , satisfying the following equation:

$$U(\psi_{in} \otimes \psi_A) = \psi(\phi_{1_0}, \dots, \phi_{n_{d-1}}) \otimes \psi_B. \quad (12)$$

where  $\psi_A \in \mathcal{H}^{d^p}$  and  $\psi_B \in \mathcal{H}^{d^{n-m+p}}$  denotes ancillary states independent of the input parameters, and  $\psi(\phi_{1_0}, \dots, \phi_{n_{d-1}}) \in \mathcal{H}^{d^m}$  is the compressed state. If such

a unitary matrix exists, we can invert the whole process to finish the retrieving process and recover the original state. Without loss of generality, we suppose  $\psi_A = |0\rangle^p$  and  $\psi_B = |0\rangle^{(n-m+p)}$ . Then, the equation (12) can be rewritten as,

$$U\left[\bigotimes_{k=1}^n \left(\sum_{j=0}^{d-1} e^{i\phi_{k,j}} |j\rangle\right) \otimes |0\rangle^{\otimes p}\right] = \psi(\phi_{1,0}, \dots, \phi_{n,d-1}) \otimes |0\rangle^{\otimes (p+n-m)}. \quad (13)$$

Clearly, the final superposition state contains no term in the form,  $|a_1 a_2 \dots a_{n+p}\rangle$ , with  $a_{m+1}, a_{m+2}, \dots, a_{n+p}$  are not all zeros. So the coefficients of the unitary matrix  $U$ ,  $u_{i_1 i_2 \dots i_{n+p}, j_1 j_2 \dots j_{n+p}}$ , must obey the following relation,

$$\sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \dots \sum_{j_n=0}^{d-1} \prod_{k=1}^n e^{i\phi_{k,j_k}} u_{a_1 \dots a_{n+p}, j_1 j_2 \dots j_n 00 \dots 0} = 0, \quad (14)$$

where, as we mentioned,  $a_{m+1}, a_{m+2}, \dots, a_{n+p}$  are not all zeros.

Lemma 1: If for any arbitrary  $\phi_{1,0} \dots \phi_{n,d-1} \in [0, 2\pi]$ ,

$$\sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \dots \sum_{j_n=0}^{d-1} \prod_{k=1}^n e^{i\phi_{k,j_k}} x_{j_1 j_2 \dots j_n} = 0, \quad (15)$$

then each of the coefficient,  $x_{j_1 \dots j_n} = 0$ .

*Proof:* We set each parameter  $\phi_{k,1}$  to one of the  $d$  possible values,  $0, \frac{2\pi}{d}, \frac{4\pi}{d}, \dots, \frac{2(d-1)\pi}{d}$ , and we set  $\phi_{k,j} = j\phi_{k,1}$  for every  $0 \leq j \leq d-1$  and every  $1 \leq k \leq n$ , then  $d^n$  equations are formed.

Now we denotes the coefficient matrix of these  $d^n$  equations as  $A_n$ . We only have to prove that  $|A_n| \neq 0$ . According to the above construction method, we have

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{d-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{d-1} & \omega^{2(d-1)} & \dots & \omega^{(d-1)^2} \end{pmatrix}, \quad (16)$$

where  $\omega = e^{\frac{2\pi i}{d}}$ . It is a Vandermonde matrix, so its determinant takes the form,

$$|A_1| = \prod_{0 \leq k < l \leq (d-1)} (\omega^k - \omega^l). \quad (17)$$

And we also have,

$$A_{n+1} = \begin{pmatrix} A_n & A_n & A_n & \dots & A_n \\ A_n & A_n \omega & A_n \omega^2 & \dots & A_n \omega^{d-1} \\ A_n & A_n \omega^2 & A_n \omega^4 & \dots & A_n \omega^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n & A_n \omega^{d-1} & A_n \omega^{2(d-1)} & \dots & A_n \omega^{(d-1)^2} \end{pmatrix} \quad (18)$$

$$|A_n| = \left[ \prod_{0 \leq k < l \leq (d-1)} (\omega^k - \omega^l) \right]^N. \quad (19)$$

While  $k < l$ ,  $\omega^k \neq \omega^l$ , hence these determinants are not zero. Q.E.D.

Coming back to the matrix  $U$  in (14), from Lemma 1, each column consists at most  $d^m$  non-zero terms, and these non-zero terms are on the same rows. We may interchange the rows and swap the columns to have the matrix  $U$  like the following form,

$$U = \begin{pmatrix} u_{a_{11}} & u_{a_{12}} & \dots & u_{a_{1d^n}} & \dots & u_{a_{1d^n+p}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ u_{a_{d^m 1}} & u_{a_{d^m 2}} & \dots & u_{a_{d^m d^n}} & \dots & u_{a_{d^m d^n+p}} \\ 0 & 0 & \dots & 0 & \dots & \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & \dots & 0 & \dots & \end{pmatrix} \quad (20)$$

It is a plain fact that if  $m < n$ ,  $|U| = 0$ . This contradicts the proposition that  $U$  is a unitary matrix. We thus conclude that the compressing and retrieving of the quantum phase information is not allowed by quantum mechanics.

*Discussions.*—As for no-cloning theorem [6], we may not clone an un-known quantum state perfectly, however, we can try to clone it approximately [11, 14–16] or probabilistically [17]. While we find that the quantum phase information can not be compressed and retrieved perfectly, it may still be interesting to design protocols for different purposes so that the quantum phase information can be compressed in some other senses.

We also would like to point out that the quantum information compressing has already been widely studied in quantum coding theories, see for example Refs.[18, 19], where the ensemble of signals from the signal source represented by density matrix is considered. However our compressing of the quantum phase information in this Letter is in a completely different framework.

A global phase of a quantum state in general cannot be detected, only the relative phase has the physical meaning, it needs at least two energy levels to encode it. This might be the simplest no-compressing of the quantum phase information. So the problem studied in this paper should be fundamental for any physical systems.

In summary, supplementary to those well-known impossibilities for quantum information theory [6–10], we show that the compressing and retrieving of quantum phase information is not allowed by quantum mechanics. Similar as for other impossibility cases, it is also expected that we may design some other phase information compressing protocols and it would also be interesting to relate our result with some other fundamental theorems in quantum information and quantum mechanics.

This work is supported by NSFC (10974247, 11175248), “973” program (2010CB922904), NFFTBS (J1030310).

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